Dynamics of a Fractional-Order Predator-Prey Model with Infectious Diseases in Prey

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Abstract

In this paper, a dynamical analysis of a fractional-order predator-prey model with infectious diseases in prey is performed. First, we prove the existence, uniqueness, non-negativity, and boundedness of the solution. We also show that the model has at most five equilibrium points, namely the origin, the infected prey and predator extinction point, the infected prey extinction point, the predator extinction point, and the co-existence point. For the first four equilibrium points, we show that the local stability properties of the fractional-order system are the same as the first-order system, but for the co-existence point, we have different local stability properties. We also present the global stability of each equilibrium points except for the origin point. We observe an interesting phenomenon, namely the occurrence of Hopf bifurcation around the co-existence equilibrium point driven by the order of fractional derivative. Moreover, we show some numerical simulations based on a predictor-corrector scheme to illustrate the result of our dynamical analysis.

Keywords: fractional-order, hopf bifurcation, infectious diseases, predator-prey, stability
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1. Introduction

Study of fractional-order differential equation becomes a popular research topic in science and engineering since various nonlinear phenomena can be described almost precisely by its ability [2], [5], [7], [10], [11], [14], [16], [31]. The main reason is that the fractional differential equation has capability to present the current state as a process that involves the history of the past states (or called the memory effects) [11], [8], [17], [23], [25], [27], [18]. Therefore, the fractional-order differential equation is gaining enormous enthusiasm from most researchers, especially in biological modeling such as ecological and epidemiological models or a combination of both which is called eco-epidemiological models [16], [18], [21], [22], [3], [20], [24], [26]. Here, we consider an eco-epidemiological model that studies the interaction between population of prey and its predator, where the prey population is assumed to grow logistically and may be infected by some microbiological organism such as pathogen or parasite. Due to the infectious diseases, we classify the prey into two compartments, namely susceptible and infected prey where the disease transmission between them obeys a bilinear incident rate. In several references, predator attacks only infected prey due to its natural instinct as in [16], [20]. But in this paper, we assume that the predator consumes both of preys because in some cases, it is difficult for predator to distinguish the susceptible and infected prey. We use Holling type-I as the predator functional response. We also consider that the growth rates of both prey and predator not only depend on the current state but also on all previous states, and thus we consider the following system of fractional order differential equations.

$$\begin{array}{rcl} D_*^\alpha x_s & = & rx_s\left(1-\frac{x_s+x_i}{K}\right)-ax_sx_i-mx_sy \\ D_*^\alpha x_i & = & ax_sx_i-bx_i-nx_iy \\ D_*^\alpha y & = & cx_sy+dx_iy-ey \end{array}, \tag{1}$$

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where $D_*^\alpha u$ denotes the Caputo fractional derivative of order α which will be introduced in the next section. Here, $x_s(\tilde{t})$, $x_i(\tilde{t})$, and $y(\tilde{t})$ denote the densities of susceptible prey population, infected prey population and predator population, respectively. Prey is growing logistically with intrinsic growth rate r and carrying capacity K. Parameter a, b, m, n, c, d and e are positive constant where a is the prey infection rate, b is death rate of infected prey, m is predation rate on susceptible prey, n is predation rate on infected prey, c, d are ratio of biomass conversion of susceptible and infected prey and e is predator death rate. Next, we simplify system (1) by variable scaling $(S, I, P, t) \to \left(\frac{x_s}{K}, \frac{x_i}{K}, \frac{my}{r}, r\tilde{t}\right)$ and obtained

$$\begin{array}{rcl} D_*^\alpha S &=& (1 - S - (1+\beta) I - P) S \\ D_*^\alpha I &=& (\beta S - \delta - \mu P) I \\ D_*^\alpha P &=& (\eta S + \omega I - \zeta) P \end{array}, \tag{2}$$

where $\beta = \frac{aK}{r}$, $\delta = \frac{b}{r}$, $\mu = \frac{n}{m}$, $\eta = \frac{cK}{r}$, $\omega = \frac{dK}{r}$, and $\zeta = \frac{e}{r}$. Note that the simplification has transformed the system (1) into a non-dimensional system (2). This means that the scale of each population density of system (2) are different from system (1), but the dynamics of system (2) are qualitatively the same as the system (1). We can also confirm that the number of parameters has been reduced so the dynamical analysis of system (2) is simpler than the previous one.

In this paper, we use Caputo fractional-order (CFO) operator with $\alpha \in (0, 1]$ as the fractional-order derivative. Due to its biological nature, i.e. the density of population is always positive, we are interested to study the solution of system (2) only in $\mathbb{R}_+^3$ for all $t \geq 0$. This paper aims to explain the dynamics of the system (2), which are arranged as follows. We first present several lemmas and theorems on fractional-order differential equation in section 2. In sections 3 and 4, we prove that the solution of the model exists and unique. We also show that the solutions are uniformly bounded and non-negative. In sections 5, 6, and 7, we investigate the equilibrium points, their existence, their local and global stability, and the existence of Hopf bifurcation. To illustrate the result from the previous section, we do some numerical simulations in section 8. We end this work with the conclusion in section 9.

2. Preliminaries

To support the theoretical study, we introduce several materials about the fractional-order differential equation that consists of definitions, lemmas, and theorems as follows.

Definition 1. (See [19]). The CFO derivative with $\alpha \in (n-1, n]$ of $f(t),\ t \geq 0$ is defined by

$$D_*^\alpha f(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

where $f^{(n)}$ represents the nth order derivative of $f(t)$, $n = \lceil \alpha \rceil$, and Γ is a Gamma function. Particularly, when $\alpha \in (0, 1]$, we have

$$D_*^\alpha f(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds.$$

Theorem 2.1. *(See [15]). Consider the following CFO system*

$$D_*^\alpha \vec{x}(t) = f(\vec{x}(t)),\ x \in \mathbb{R}^n,\ n \in \mathbb{N},\ \vec{x}(0) \geq 0,\ \textit{and } \alpha \in (0, 1]. \tag{3}$$

A point $\vec{x}^*$ that satisfies $\vec{f}(\vec{x}^*) = 0$ is called the equilibrium point. It is locally asymptotically stable if its Jacobian matrix $J = \frac{\partial f}{\partial x}$ evaluated at $\vec{x}^*$ provide the eigenvalues that satisfy $|\arg(\lambda_j)| > \frac{\alpha\pi}{2}$ for all $j \in n$.

Lemma 2.2. *(See [13]). Consider the following CFO system*

$$D_*^\alpha x(t) = f(t, x),\ x(0) \geq 0,\ \alpha \in (0, 1],\ f : [0, \infty) \times \Omega \to \mathbb{R}^n,\ \Omega \in \mathbb{R}^n. \tag{4}$$

Then there exists a unique solution of system (4) on $[0, \infty) \times \Omega$ if $f(t, x)$ satisfies the locally Lipschitz condition with respect to x.

Lemma 2.3. *(See [11]). Let $u(t)$ be a continuous function on $[0, +\infty)$ and satisfy*

$$D_*^\alpha u(t) \leq -\lambda u(t) + \mu,\ (\lambda, \mu) \in \mathbb{R}^2,\ \lambda \neq 0,\ u(0) = u_0 \geq 0,\ \textit{and } \alpha \in (0, 1]. \tag{5}$$

Then the solution of (5) has the form

$$u(t) \leq \left(u_0 - \frac{\mu}{\lambda}\right) E_\alpha\left[-\lambda t^\alpha\right] + \frac{\mu}{\lambda}.$$

Lemma 2.4. *(See [28]). Let $x(t) \in \mathbb{R}_+$ be a continuous and derivable function. Then, for any time instant $t \geq t_0$*

$$D_*^\alpha\left[x(t) - x^* - x^* \ln \frac{x(t)}{x^*}\right] \leq \left(1 - \frac{x^*}{x(t)}\right),\ x^* \in \mathbb{R}_+,\ \forall \alpha \in (0, 1].$$

Lemma 2.5. *(See [9]). Suppose D is a bounded closet set. Every solution of $D_*^\alpha x(t) = f(x)$ starts from a point in D and remains in D for all time, If $V(x) : D \to \mathbb{R}$ with continuous first partial derivatives satisfies $D_*^\alpha \leq 0$. Let $E = \{x \,|D_*^\alpha V = 0\}$ and M be the largest invariant set of E. Then every solution $x(t)$ originating in D tends to M as $t \to \infty$. Particularly, when $M = \{0\}$, then $x \to 0$, $t \to \infty$.*

3. Existence and uniqueness

In this section, we prove that the solution of system (2) is exists and unique, which is shown by the following theorem.

Theorem 3.1. *Consider system (2) with initial condition $S_{t_0} \geq 0$, $I_{t_0} \geq 0$ $P_{t_0} \geq 0$ and $\alpha \in (0, 1]$, $f : [t_0, \infty) \times \Omega_M \to \mathbb{R}^3$, where $\Omega_M := \left\{(S, I, P) \in \mathbb{R}_+^3 : \max\{|S|, |I|, |P| \leq M\}\right\}$ for sufficiently large M. This system* (2) *IVP has a unique solution.*

Proof: Consider a mapping $H(Z) = (H_1(Z), H_2(Z), H_3(Z))$ with

$$\begin{array}{lcl} H_1(Z) & = & (1 - S - (1+\beta) I - P) S \\ H_2(Z) & = & (\beta S - \delta - \mu P) I \\ H_3(Z) & = & (\eta S + \omega I - \zeta) P \end{array}. \tag{6}$$

For any $Z = (S, I, P)$, $\bar{Z} = \left(\bar{S}, \bar{I}, \bar{P}\right)$, $Z, \bar{Z} \in \Omega_M$, it follows from (6) that

$$\begin{array}{rcl} ||H(Z) - H(\bar{Z})|| & = & \left|H_1(Z) - H_1(\bar{Z})\right| + \left|H_2(Z) - H_2(\bar{Z})\right| + \left|H_3(Z) - H_3(\bar{Z})\right| \\ & = & \left|(S - \bar{S}) - (S^2 - \bar{S}^2) - (1+\beta)(SI - \bar{S}\bar{I}) - (SP - \bar{S}\bar{P})\right| + \\ & & \left|\beta(SI - \bar{S}\bar{I}) - \delta(I - \bar{I}) - \mu(IP - \bar{I}\bar{P})\right| + \\ & & \left|\eta(SP - \bar{S}\bar{P}) + \omega(IP - \bar{I}\bar{P}) - \zeta(P - \bar{P})\right| \\ & \leq & \left|S - \bar{S}\right| + \left|S^2 - \bar{S}^2\right| + (1+\beta)\left|SI - \bar{S}\bar{I}\right| + \left|SP - \bar{S}\bar{P}\right| + \\ & & \beta\left|SI - \bar{S}\bar{I}\right| + \delta\left|I - \bar{I}\right| + \mu\left|IP - \bar{I}\bar{P}\right| + \eta\left|SP - \bar{S}\bar{P}\right| + \\ & & \omega\left|IP - \bar{I}\bar{P}\right| + \zeta\left|P - \bar{P}\right| \\ & \leq & (1 + 4M + 2\beta M + \eta M)\left|S - \bar{S}\right| + ((1 + 2\beta + \mu + \omega)M + \delta)\left|I - \bar{I}\right| \\ & & ((1 + \eta + \mu + \omega)M + \zeta)\left|P - \bar{P}\right| \\ & \leq & L\left|\left|Z - \bar{Z}\right|\right| \end{array},$$

where $L = \max\{1 + (4 + 2\beta + \eta)M, \delta + (1 + 2\beta + \mu + \omega)M, \zeta + (1 + \eta + \mu + \omega)M\}$. Thus, the Lipschitz condition with respect to Z is satisfied by $H(Z)$. According to Lemma 2.2, that there exists a unique solution $Z(t) \in \Omega_M$ of system (2) with initial condition $Z_{t_0} = (S_{t_0}, I_{t_0}, P_{t_0})$. ■

4. Boundedness and Non-Negativity

Now, we will prove that the solutions of system (2) with the initial values start in $\mathbb{R}_+^3$ are bounded and non-negative to ensure that the biological significance is reached.

Theorem 4.1. *Consider system (2) with initial condition $S_{t_0} \geq 0$, $I_{t_0} \geq 0$ $P_{t_0} \geq 0$. Then all solutions are uniformly bounded and non-negative.*

Proof: First, we prove that if the initial condition of system (2) is non-negative then all solutions are uniformly bounded. Define a function $V(t)=\left(\frac{\mu\eta}{\omega}+\frac{2\beta}{1+\beta}\right)S+2I+\frac{\mu}{\omega}P$, then we have

$$\begin{aligned} D_*^\alpha V(t)+\xi V(t) &= \left(\tfrac{\mu\eta}{\omega}+\tfrac{2\beta}{1+\beta}\right)(1-S-(1+\beta)I-P)S+2(\beta S-\delta-\mu P)I \\ &\quad +\tfrac{\mu}{\omega}(\eta S+\omega I-\zeta)P+\left(\tfrac{\mu\eta}{\omega}+\tfrac{2\beta}{1+\beta}\right)\xi S+2\xi I+\tfrac{\mu}{\omega}\xi P \\ &= \left(\tfrac{\mu\eta}{\omega}+\tfrac{2\beta}{1+\beta}\right)S+\left(\tfrac{\mu\eta}{\omega}+\tfrac{2\beta}{1+\beta}\right)\xi S-\left(\tfrac{\mu\eta}{\omega}+\tfrac{2\beta}{1+\beta}\right)S^2-\tfrac{(1+\beta)\mu\eta}{\omega}SI \\ &\quad -\tfrac{2\beta}{1+\beta}SP-\mu IP+2(\xi-\delta)I+(\xi-\zeta)\tfrac{\mu}{\omega}P. \end{aligned}$$

Choose $\xi<\min\{\delta,\zeta\}$ then

$$\begin{aligned} D_*^\alpha V(t)+\xi V(t) &\le \left(\tfrac{\mu\eta}{\omega}+\tfrac{2\beta}{1+\beta}\right)S+\left(\tfrac{\mu\eta}{\omega}+\tfrac{2\beta}{1+\beta}\right)\xi S-\left(\tfrac{\mu\eta}{\omega}+\tfrac{2\beta}{1+\beta}\right)S^2 \\ &= -\left(\tfrac{\mu\eta}{\omega}+\tfrac{2\beta}{1+\beta}\right)\left(S-\tfrac{1+\xi}{2}\right)^2+\left(\tfrac{\mu\eta}{\omega}+\tfrac{2\beta}{1+\beta}\right)\left(\tfrac{1+\xi}{2}\right)^2 \\ &\le \left(\tfrac{\mu\eta}{\omega}+\tfrac{2\beta}{1+\beta}\right)\left(\tfrac{1+\xi}{2}\right)^2. \end{aligned}$$

By Lemma (2.3), we have

$$V(t)\le\left(V(0)-\frac{1}{\xi}\left(\frac{\mu\eta}{\omega}+\frac{2\beta}{1+\beta}\right)\left(\frac{1+\xi}{2}\right)^2\right)E_\alpha\left[-\xi(t)^\alpha\right]+\frac{1}{\xi}\left(\frac{\mu\eta}{\omega}+\frac{2\beta}{1+\beta}\right)\left(\frac{1+\xi}{2}\right)^2.$$

Notice that $V(t)\to\frac{1}{\xi}\left(\frac{\mu\eta}{\omega}+\frac{2\beta}{1+\beta}\right)\left(\frac{1+\xi}{2}\right)^2$ for $t\to\infty$. Therefore, for non-negative initial condition involve all solutions of system (2) are confined to the region Ω, where

$$\Omega:=\left\{(S,I,P)\in\mathbb{R}_+^3\ :\ V(t)\le\frac{1}{\xi}\left(\frac{\mu\eta}{\omega}+\frac{2\beta}{1+\beta}\right)\left(\frac{1+\xi}{2}\right)^2+\varepsilon,\ \varepsilon>0\right\}. \tag{7}$$

Next we prove that if the initial condition is non-negative, then all solutions are non-negative. From inequality (7) we have that

$$\left(\frac{\mu\eta}{\omega}+\frac{2\beta}{1+\beta}\right)S+2I+\frac{\mu}{\omega}P\le\frac{1}{\xi}\left(\frac{\mu\eta}{\omega}+\frac{2\beta}{1+\beta}\right)\left(\frac{1+\xi}{2}\right)^2. \tag{8}$$

Based on equation (2) and inequality (8) we get

$$\begin{aligned} D_*^\alpha S &\ge \left[1-\tfrac{1}{\xi}\left(\tfrac{1+\xi}{2}\right)^2-\tfrac{1+\beta}{2\xi}\left(\tfrac{\mu\eta}{\omega}+\tfrac{2\beta}{1+\beta}\right)\left(\tfrac{1+\xi}{2}\right)^2-\tfrac{\omega}{\mu\xi}\left(\tfrac{\mu\eta}{\omega}+\tfrac{2\beta}{1+\beta}\right)\left(\tfrac{1+\xi}{2}\right)^2\right]S \\ &= \left[1-\left(\tfrac{1}{\xi}+\left(\tfrac{1+\beta}{2\xi}+\tfrac{\omega}{\mu\xi}\right)\left(\tfrac{\mu\eta}{\omega}+\tfrac{2\beta}{1+\beta}\right)\right)\left(\tfrac{1+\xi}{2}\right)^2\right]S \\ &= \sigma_1 S,\ \text{where } \sigma_1=1-\left(\tfrac{1}{\xi}+\left(\tfrac{1+\beta}{2\xi}+\tfrac{\omega}{\mu\xi}\right)\left(\tfrac{\mu\eta}{\omega}+\tfrac{2\beta}{1+\beta}\right)\right)\left(\tfrac{1+\xi}{2}\right)^2 \end{aligned}.$$

From the standard comparison theorem for fractional-order differential equation [4] and the positivity of Mittag-Leffler function $E_{\alpha,1}(t)>0$ [29], we get $S(t)\ge S_{t_0}E_{\alpha,1}(\sigma_1 t^\alpha)$, thus we get

$$S(t)\ge 0. \tag{9}$$

Next, from the second equation in system (2), inequality (8) and (9) we obtain

$$\begin{aligned} D_*^\alpha I &\ge \left[\beta S-\delta-\tfrac{\omega}{\xi}\left(\tfrac{\mu\eta}{\omega}+\tfrac{2\beta}{1+\beta}\right)\left(\tfrac{1+\xi}{2}\right)^2\right]I \\ &\ge -\left[\delta+\tfrac{\omega}{\xi}\left(\tfrac{\mu\eta}{\omega}+\tfrac{2\beta}{1+\beta}\right)\left(\tfrac{1+\xi}{2}\right)^2\right]I \\ &= -\sigma_2 I,\ \text{where } \sigma_2=\delta+\tfrac{\omega}{\xi}\left(\tfrac{\mu\eta}{\omega}+\tfrac{2\beta}{1+\beta}\right)\left(\tfrac{1+\xi}{2}\right)^2 \end{aligned}.$$

Therefore $I(t) \geq I_{t_0} E_{\alpha,1}(-\sigma_2 t^\alpha)$, thus we have

$$I(t) \geq 0. \tag{10}$$

Last, from the third equation in system (2), inequality (9) and (10) we obtain

$$\begin{array}{rcl} D_*^\alpha P & \geq & (\eta S + \omega I - \zeta)\, P \\ & \geq & -\zeta P \end{array}.$$

Therefore $P(t) \geq P_{t_0} E_{\alpha,1}(-\zeta t^\alpha)$, thus we have $P_{t_0} \geq 0$, and Theorem 4.1 is completely proven. ■

5. Equilibrium Points and their Local Stability

To investigate the dynamical behavior of system (2), we identify the equilibrium points, their existence and analyze their stability. According to the Theorem (2.1), the equilibrium points is obtained by solving the simultaneous equations:

$$\begin{array}{rcl} (1 - S - (1+\beta)\, I - P)\, S & = & 0 \\ (\beta S - \delta - \mu P)\, I & = & 0 \\ (\eta S + \omega I - \zeta)\, P & = & 0 \end{array}. \tag{11}$$

From equations (11), we have five equilibrium points for system (2) as follows:

(i) The origin point $E_0 = (0,0,0)$ which always exists.

(ii) The predator and infected prey extinction point $E_1 = (1,0,0)$, which always exists.

(iii) The infected prey extinction point $E_2 = \left(\frac{\zeta}{\eta}, 0, \frac{\eta-\zeta}{\eta}\right)$ which exists if $\eta > \zeta$.

(iv) The predator extinction point $E_3 = \left(\frac{\delta}{\beta}, \frac{\beta-\delta}{\beta(1+\beta)}, 0\right)$ which exists if $\beta > \delta$.

(v) The co-exsistence point $E^* = \left(\varphi, \frac{\zeta-\eta\varphi}{\omega}, \frac{\beta\varphi-\delta}{\mu}\right)$ where $\varphi = \frac{(\mu+\delta)\omega-(1+\beta)\mu\zeta}{(\mu+\beta)\omega-(1+\beta)\mu\eta}$ which exists if $\frac{\delta}{\beta} < \varphi < \frac{\zeta}{\eta}$ and $\omega > \mu(1+\beta)\max\left\{\frac{\zeta}{\mu+\delta}, \frac{\eta}{\mu+\beta}\right\}$ or $\omega < \mu(1+\beta)\min\left\{\frac{\zeta}{\mu+\delta}, \frac{\eta}{\mu+\beta}\right\}$.

The local stability of these equilibrium points are explained in the following Theorems.

Theorem 5.1. (i) *The origin E_0 is always a saddle point.*

(ii) *If $\beta < \delta$ and $\eta < \zeta$ then E_1 is locally asymptotically stable.*

(iii) *If $\eta > \frac{\mu+\beta}{\mu+\delta}\zeta$ then E_2 is locally asymptotically stable.*

(iv) *If $\omega < \frac{(1+\beta)(\beta\zeta-\delta\eta)}{\beta-\delta}$ then E_3 is locally asymptotically stable.*

Proof:

(i) Firstly, we identify the Jacobian matrix $J(E_0)$ and acquired

$$J(E_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\delta & 0 \\ 0 & 0 & -\zeta \end{bmatrix},$$

and gives eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -\delta$ and $\lambda_3 = -\zeta$. Thus $|\arg(\lambda_1)| = 0 < \frac{\alpha\pi}{2}$ and $|\arg(\lambda_{2,3})| = \pi > \frac{\alpha\pi}{2}$. Therefore E_0 is always a saddle point.

(ii) We compute the Jacobian matrix $J(E_1)$ and obtain

$$J(E_1) = \begin{bmatrix} -1 & -(1+\beta) & -1 \\ 0 & \beta-\delta & 0 \\ 0 & 0 & \eta-\zeta \end{bmatrix},$$

where its eigenvalues are $\lambda_1 = -1$, $\lambda_2 = \beta-\delta$ and $\lambda_3 = \eta-\zeta$. note that for λ_1 gives$|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$, but $|\arg(\lambda_{2,3})| = \pi > \frac{\alpha\pi}{2}$ if $\beta < \delta$ and $\eta < \zeta$. Consequently, E_1 become locally asymptotically stable.

(iii) Now, we determine the Jacobian matrix $J(E_2)$ and achieve

$$J(E_2) = \begin{bmatrix} -\frac{\zeta}{\eta} & -\frac{(1+\beta)\zeta}{\eta} & -\frac{\zeta}{\eta} \\ 0 & \frac{\beta\zeta-\delta\eta-(\eta-\zeta)\mu}{\eta} & 0 \\ \eta-\zeta & \frac{(\eta-\zeta)\omega}{\eta} & 0 \end{bmatrix}.$$

The corresponding eigenvalues are $\lambda_1 = \frac{\beta\zeta-\delta\eta-(\eta-\zeta)\mu}{\eta}$ and $\lambda_{2,3} - \frac{-\zeta\pm\sqrt{\zeta^2-4(\eta-\zeta)\eta\zeta}}{2\eta}$. Note that if $\eta > \frac{\mu+\beta}{\mu+\delta}\zeta$ then $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$. It is also clear that $\lambda_{2,3}$ always satisfy $|\arg(\lambda_{2,3})| > \frac{\alpha\pi}{2}$. Hence, we have Theorem 5.1.(iii).

(iv) Lastly, we investigate $J(E_3)$ and get

$$J(E_3) = \begin{bmatrix} -\frac{\delta}{\beta} & -\frac{(1+\beta)\delta}{\beta} & -\frac{\delta}{\beta} \\ \frac{\beta-\delta}{1+\beta} & 0 & -\frac{\mu(\beta-\delta)}{\beta(1+\beta)} \\ 0 & 0 & \frac{\delta\eta-\beta\zeta}{\beta}+\frac{\omega(\beta-\delta)}{\beta(1+\beta)} \end{bmatrix},$$

where its eigenvalues are $\lambda_1 = \frac{\delta\eta-\beta\zeta}{\beta}+\frac{\omega(\beta-\delta)}{\beta(1+\beta)}$ and $\lambda_{2,3} = \frac{-\delta\pm\sqrt{\delta^2-4(\beta-\delta)\beta\delta}}{2\beta}$. If $\omega < \frac{(1+\beta)(\beta\zeta-\delta\eta)}{\beta-\delta}$ then $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$. Furthermore, it can be easily be proven that $|\arg(\lambda_{2,3})| > \frac{\alpha\pi}{2}$ is always fullfilled.

We can observe that the eigenvalues of $J(E_0)$ and $J(E_1)$ are always real numbers. Therefore, the order-α has no effect to their stability. Furthermore, if the stability conditions for E_2 and E_3 are satisfied, then the Jacobian matrices $J(E_2)$ and $J(E_3)$ have always eigenvalues where their real parts are negatives. Hence, the eigenvalues always satisfy $|\arg(\lambda)| > \frac{\alpha\pi}{2}, \forall\alpha \in (0,1]$. We conclude that the stability properties of these equilibrium points are exactly the same as for the case of integer-order model. ■

Theorem 5.2. *Suppose that:*

$$\begin{array}{rcl} \mu^* &=& \frac{\omega}{\beta}\left(\frac{\beta\varphi-\delta}{\zeta-\eta\varphi}\right)\left(\frac{\beta\zeta-(1+\beta)\eta\varphi}{(1+\beta)\varphi+(\beta\varphi-\delta)\eta}\right) \\ \xi_1 &=& \frac{(\beta\varphi-\delta)\eta\omega\varphi+(1+\beta)(\zeta-\eta\varphi)\beta\mu\varphi}{\mu\omega} \\ \xi_2 &=& \frac{(\zeta-\eta\varphi)(\beta\varphi-\delta)(\omega-\eta\mu)\beta\varphi}{\mu\omega} \\ D(P) &=& 18\varphi\xi_1\xi_2+(\varphi\xi_1)^2-4\xi_2(\varphi)^3-4(\xi_1)^3-27(\xi_2)^2 \end{array}.$$

E^* is called locally asymptotically stable if one of the following statements is satisfied.

(i) $D(P) > 0$ *and* $\mu^* < \mu < \frac{\omega}{\eta}$, *or;*

(ii) $D(P) < 0$ *and:*

(ii.a) $\mu < \frac{\omega}{\eta}$ *and* $0 < \alpha < \frac{2}{3}$.

(ii.b) $\mu = \mu^*$

Proof: By computing the Jacobian matrix $J(E^*)$, we obtain

$$J(E^*) = \begin{bmatrix} -\varphi & -(1+\beta)\varphi & -\varphi \\ \frac{(\zeta-\eta\varphi)\beta}{\omega} & 0 & -\frac{(\zeta-\eta\varphi)\mu}{\omega} \\ \frac{(\beta\varphi-\delta)\eta}{\mu} & \frac{(\beta\varphi-\delta)\omega}{\mu} & 0 \end{bmatrix}.$$

This Jacobian matrix gives polynomial characteristic: $P = \lambda^3+\varphi\lambda^2+\xi_1\lambda+\xi_2 = 0$. By using Routh-Hurwitz condition for fractional-order dynamical system (see Proposition 1 in [1]), the local stability condition of co-existence point E^* is proven. ■

6. Global Stability

This section presents about the global stability of equilibrium points which are described by these folowing theorems.

Theorem 6.1. *E_1 is globally asymptotically stable if* $\omega < \frac{(1+\beta)\mu\eta}{\beta}$, $\beta < \delta$, *and* $\eta < \zeta$.

Proof: We first define a Lyapunov function as follows.

$$V(S,I,P) = (S-1-\ln S)+\frac{1+\beta}{\beta}I+\frac{1}{\eta}P.$$

For the initial analysis, we investigate that $V(E_1) = 0$, so that the first requirement is satisfied. Furthermore, by applying Lemma 2.4 we obtain

$$\begin{array}{rcl} D_*^\alpha V(S,I,P) & \leq & (S-1)(1-S-(1+\beta)I-P)+\frac{1+\beta}{\beta}(\beta S-\delta-\mu P)I+\frac{1}{\eta}(\eta S+\omega I-\zeta)P \\ & = & -(S-1)^2+(1+\beta)I+P-\frac{1+\beta}{\beta}\delta I-\frac{1+\beta}{\beta}\mu IP+\frac{\omega}{\eta}IP-\frac{\zeta}{\eta}P \\ & = & -(S-1)^2-\left(\frac{\delta}{\beta}-1\right)(1+\beta)I-\left(\frac{\zeta}{\eta}-1\right)P-\left(\frac{(1+\beta)\mu}{\beta}-\frac{\omega}{\eta}\right)IP \\ & \leq & 0 \end{array}.$$

By following Lemma 2.5, thus every non-negative solution tends to E_1 which means that the equilibrium point E_1 is globally asymptotically stable. ■

Theorem 6.2. *E_2 is globally asymptotically stable if* $\frac{\beta\zeta-\delta\eta}{\eta-\zeta} < \frac{\beta\omega}{(1+\beta)\eta} < \mu$.

Proof: To proof the global stability of E_2, we construct a Lyapunov function

$$V(S,I,P)=\left(S-\frac{\zeta}{\eta}-\frac{\zeta}{\eta}\ln\frac{\eta S}{\zeta}\right)+\frac{1+\beta}{\beta}I+\frac{1}{\eta}\left(P-\frac{\eta-\zeta}{\eta}-\frac{\eta-\zeta}{\eta}\ln\frac{\eta P}{\eta-\zeta}\right).$$

We can confirm that $V(E_2) = 0$ which indicates the first condition is fulfilled. Now, by using Lemma 2.4 we show that

$$\begin{array}{rcl} D_*^\alpha V(S,I,P) & \leq & \left(S-\frac{\zeta}{\eta}\right)(1-S-(1+\beta)I-P)+\frac{1+\beta}{\beta}(\beta S-\delta-\mu P)I \\ & & +\frac{1}{\eta}\left(P-\frac{\eta-\zeta}{\eta}\right)(\eta S+\omega I-\zeta) \\ & = & -\left(S-\frac{\zeta}{\eta}\right)^2-(1+\beta)\left(S-\frac{\zeta}{\eta}\right)I-\left(S-\frac{\zeta}{\eta}\right)\left(P-\frac{\eta-\zeta}{\eta}\right) \\ & & +(1+\beta)SI-\frac{(1+\beta)\delta}{\beta}I-\frac{(1+\beta)\mu}{\beta}IP \\ & & +\left(P-\frac{\eta-\zeta}{\eta}\right)S+\left(P-\frac{\eta-\zeta}{\eta}\right)\frac{\omega}{\eta}I-\left(P-\frac{\eta-\zeta}{\eta}\right)\frac{\zeta}{\eta} \\ & = & -\left(S-\frac{\zeta}{\eta}\right)^2-\left(\frac{(\eta-\zeta)\omega}{\eta^2}-\frac{(1+\beta)(\beta\zeta-\delta\eta)}{\beta\eta}\right)I-\left(\frac{(1+\beta)\mu}{\beta}-\frac{\omega}{\eta}\right)IP \\ & \leq & 0 \end{array}.$$

According to Lemma 2.5, it is found that every non-negative solution tends to E_2 so that the globally asymptotically stable of equilibrium point E_2 is achieved. ■

Theorem 6.3. *E_3 is globally asymptotically stable if* $\frac{\beta^2\zeta}{(\beta-\delta)\mu+\beta\delta} < \eta < \frac{\beta\omega}{(1+\beta)\mu}$.

Proof: A Lyapunov function is defined as

$$V(S,I,P)=\left(S-\frac{\delta}{\beta}-\frac{\delta}{\beta}\ln\frac{\beta S}{\delta}\right)+\frac{1+\beta}{\beta}\left(I-\frac{\beta-\delta}{\beta(1+\beta)}-\frac{\beta-\delta}{\beta(1+\beta)}\ln\frac{\beta(1+\beta)I}{\beta-\delta}\right)+\frac{1}{\eta}P.$$

We check that $V(E_3) = 0$ so that the Lyapunov function suitable with the expected conditions. We apply Lemma 2.4 to the Lyapunov function and obtain

$$\begin{array}{rcl} D_*^\alpha V(S,I,P) & \leq & \left(S-\frac{\delta}{\beta}\right)(1-S-(1+\beta)I-P)+\frac{1+\beta}{\beta}\left(I-\frac{\beta-\delta}{\beta(1+\beta)}\right)(\beta S-\delta-\mu P) \\ & & +\frac{1}{\eta}(\eta S+\omega I-\zeta)P \\ & = & -\left(S-\frac{\delta}{\beta}\right)^2-(1+\beta)\left(I-\frac{\beta-\delta}{\beta(1+\beta)}\right)\left(S-\frac{\delta}{\beta}\right)-\left(S-\frac{\delta}{\beta}\right)P \\ & & +\left(I-\frac{\beta-\delta}{\beta(1+\beta)}\right)\left((1+\beta)S-\frac{(1+\beta)\delta}{\beta}-\frac{(1+\beta)\mu}{\beta}P\right)+SP+\frac{\omega}{\eta}IP-\frac{\zeta}{\eta}P \\ & = & -\left(S-\frac{\delta}{\beta}\right)^2-\left(\frac{\zeta}{\eta}-\frac{(\beta-\delta)\mu+\beta\delta}{\beta^2}\right)P-\left(\frac{(1+\beta)\mu}{\beta}-\frac{\omega}{\eta}\right)IP \end{array}.$$

By using Lemma 2.5, we conclude that every non-negative solution tends to E_3 so that the globally asymptotically stable of equilibrium point E_3 is accomplished. ■

Theorem 6.4. *Suppose that:*

$$\begin{array}{rcl} 0< & \xi & <\min\{\delta,\zeta\} \\ & \theta & =\frac{1}{\xi}\left(\frac{\mu\eta}{\omega}+\frac{2\beta}{1+\beta}\right)\left(\frac{1+\xi}{2}\right)^2 \\ \frac{2(\zeta-\eta\varphi)(\beta\varphi-\delta)}{2(\zeta-\eta\varphi)\theta+(\beta\varphi-\delta)\theta}< & \omega & <\frac{(1+\beta)\mu\eta}{\beta} \end{array},$$

then the co-existence point E^* is globally asymptotically stable.

Proof: Suppose that $E^*=(S^*,I^*,P^*)$ is the co-existence point. We define a Lyapunov function by

$$V(S,I,P)=\left(S-S^*-S^*\ln\frac{S}{S^*}\right)+a_1\left(I-I^*-I^*\ln\frac{I}{I^*}\right)+a_2\left(P-P^*-P^*\ln\frac{P}{P^*}\right).$$

It is clear that $V(E^*)=0$. Now, by using Lemma 2.4 we have

$$\begin{array}{rcl} D_*^\alpha V(S,I,P) & \leq & (S-S^*)(1-S-(1+\beta)I-P)+a_1(I-I^*)(\beta S-\delta-\mu P) \\ & & +a_2(P-P^*)(\eta S+\omega I-\zeta) \\ & = & -(S-S^*)((S-S^*)+(1+\beta)(I-I^*)+(P-P^*)) \\ & & +a_1(I-I^*)(\beta(S-S^*)-\mu(P-P^*)) \\ & & +a_2(P-P^*)(\eta(S-S^*)+\omega(I-I^*)) \\ & = & -(S-S^*)^2-(1+\beta)(S-S^*)(I-I^*)-(S-S^*)(P-P^*) \\ & & +a_1\beta(S-S^*)(I-I^*)-a_1\mu(I-I^*)(P-P^*) \\ & & +a_2\eta(S-S^*)(P-P^*)+a_2\omega(I-I^*)(P-P^*) \\ & = & -(S-S^*)^2-((1+\beta)-a_1\beta)(S-S^*)(I-I^*) \\ & & -(1-a_2\eta)(S-S^*)(P-P^*)-(a_1\mu-a_2\omega)(I-I^*)(P-P^*) \end{array}.$$

Choose $a_1=\frac{1+\beta}{\beta}$ and $a_2=\frac{1}{\eta}$ then we get:

$$\begin{array}{rcl} D_*^\alpha V(S,I,P) & \leq & -(S-S^*)^2-\left(\frac{(1+\beta)\mu}{\beta}-\frac{\omega}{\eta}\right)(I-I^*)(P-P^*) \\ & \leq & -\left(\frac{(1+\beta)\mu}{\beta}-\frac{\omega}{\eta}\right)(I-I^*)(P-P^*) \\ & = & -\left(\frac{(1+\beta)\mu}{\beta}-\frac{\omega}{\eta}\right)IP+\left(\frac{(1+\beta)\mu}{\beta}-\frac{\omega}{\eta}\right)I^*P \\ & & +\left(\frac{(1+\beta)\mu}{\beta}-\frac{\omega}{\eta}\right)P^*I-\left(\frac{(1+\beta)\mu}{\beta}-\frac{\omega}{\eta}\right)I^*P^* \end{array}.$$

Because $\omega<\frac{(1+\beta)\mu\eta}{\beta}$, we have

$$\begin{array}{rcl} D_*^\alpha V(S,I,P) & \leq & \left(\frac{(1+\beta)\mu}{\beta}-\frac{\omega}{\eta}\right)I^*P+\left(\frac{(1+\beta)\mu}{\beta}-\frac{\omega}{\eta}\right)P^*I-\left(\frac{(1+\beta)\mu}{\beta}-\frac{\omega}{\eta}\right)I^*P^* \\ & = & -\left(\frac{(1+\beta)\mu}{\beta}-\frac{\omega}{\eta}\right)(I^*P^*-I^*P-P^*I) \end{array}. \tag{12}$$

From inequality (8) we obtain $I\leq\frac{\theta}{2}$ and $P\leq\frac{\omega\theta}{\mu}$ so that the inequality (12) becomes

$$D_*^\alpha V(S,I,P)<-\left(\frac{(1+\beta)\mu}{\beta}-\frac{\omega}{\eta}\right)\left(I^*P^*-\frac{\omega\theta}{\mu}I^*-\frac{\theta}{2}P^*\right). \tag{13}$$

By Subtituting $E^*=\left(\varphi,\frac{\zeta-\eta\varphi}{\omega},\frac{\beta\varphi-\delta}{\mu}\right)$ into inequality (13), we obtain

$$D_*^\alpha V(S,I,P)<-\left(\frac{(1+\beta)\mu}{\beta}-\frac{\omega}{\eta}\right)\left(\frac{(\zeta-\eta\varphi)(\beta\varphi-\delta)}{\mu\omega}-\frac{2(\zeta-\eta\varphi)\theta+(\beta\varphi-\delta)\theta}{2\mu}\right)\leq 0. \tag{14}$$

By using the same manner with Huo et al, we apply Lemma 2.5 to confirm that every non-negative solution tends to E^* so that we completely proof that E^* is globally asymptotically stable. ■

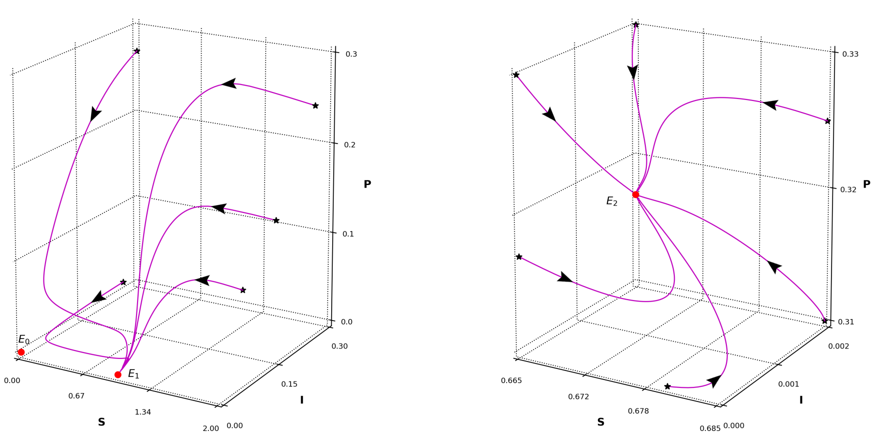
7. EXISTENCE OF HOPF BIFURCATION

Now we will demonstrate that there exists a Hopf bifurcation in system (2) at the co-existence equilibrium point E^* where α is the bifurcation parameter. This bifurcation arises if a stable focus equilibrium point changes to unstable focus and the limit cycle occurs simultaneously when a bifurcation parameter is variated. From the Jacobian matrix $J(E^*)$, we have polynomial characteristic $\lambda^3+\varphi\lambda^2+\xi_1\lambda+\xi_2=0$. By applying Cardano's formula [30], we obtain the eigenvalues defined by $\lambda_1=U+T-\frac{\varphi}{3}$ and $\lambda_2=\theta\pm\omega i$ where

$$\begin{aligned} \theta &= -\tfrac{U+T}{2}-\tfrac{\varphi}{3}, & w &= \tfrac{-1+\sqrt{3}i}{2}, \\ \omega &= \tfrac{(U-T)\sqrt{3}}{2}, & Q &= \tfrac{3\xi_1-\varphi^2}{9}, \\ U &= w\sqrt[3]{R+\sqrt{R^2+Q^3}}, & R &= \tfrac{9\xi_1\varphi-27\xi_2-2\varphi^3}{54}, \\ T &= w^2\sqrt[3]{R-\sqrt{R^2+Q^3}}. \end{aligned}$$

Suppose that $\{U,T\}\in\mathbb{R}$ and $3(U+T)<\varphi<-\frac{3(U+T)}{2}$ so that the eigenvalues are $\lambda_1<0$ and a pair of complex conjugate $\lambda_{2,3}=\theta\pm\omega i$ where $\theta>0$. It is easy to confirm that $m(\alpha^*)=0$ and $\left.\frac{dm(\alpha)}{d\alpha}\right|_{\alpha=\alpha^*}=\frac{\pi}{2}$ with $m(\alpha)=\frac{\alpha\pi}{2}-\min_{1\le i\le 3}|\arg(\lambda_i)|$ and $\alpha^*=\frac{2}{\pi}|\arg(\lambda_{2,3})|$. Based on Theorem 3 in [12], a Hopf bifurcation occurs around the equilibrium point E^* when α passes through α^*. Consequently, we have the following theorem:

Theorem 7.1. *Suppose that $U\in\mathbb{R}$, $T\in\mathbb{R}$ and $3(U+T)<\varphi<-\frac{3(U+T)}{2}$. The equilibrium point E^* undergoes a Hopf bifurcation when α passes through $\alpha^*=\frac{2}{\pi}|\arg(\lambda_{2,3})|$.*

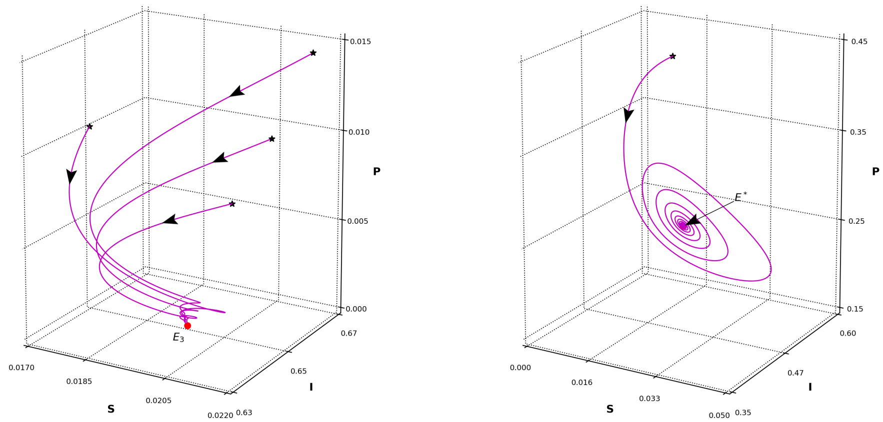


(a) $\eta=0.13$ (b) $\eta=0.65$

Figure 1: 3-D Phaseportraits of system (2) with parameter: $\beta=0.51$, $\delta=0.52$, $\mu=0.01$, $\omega=0.92$, $\zeta=0.44$, $\alpha=0.8$ and time step $\Delta t=0.1$

8. NUMERICAL SIMULATIONS

To verify the previous theoritical results, numerical simulations are performed by using predictor-corrector approach of fractional-order differential equation [6]. Since the field data is not available, we use hypothetical parameter values which are satisified the stability conditions from the previous analytical studies. We first set the parameter values as follows: $\beta=0.51$, $\delta=0.52$, $\mu=0.01$, $\eta=0.13$, $\omega=0.92$, $\zeta=0.44$ and $\alpha=0.8$. Here, we only have two equilibrium point, i.e. a saddle point $E_0=(0,0,0)$ and an asymptotically (both locally and globally) stable $E_1=(1,0,0)$ (see Figure 1a). When the ratio of biomass conversion of susceptible prey is raised to $\eta = 0.65$, the stable infected prey point $E_2 = (0.677, 0, 0.323)$ appears, and the E_1 becomes a saddle point, which fit to Theorem 2.1(ii) and 6.2. This shows that the susceptible prey and predator population are maintained, and the disease infection in prey is stopped. See Figure 1b.



(a) $\omega = 0.55$ (b) $\omega = 0.92$

Figure 2: 3-D Phaseportraits of system (2) with parameter: $\beta = 0.51$, $\delta = 0.01$, $\mu = 0.01$, $\eta = 0.13$, $\zeta = 0.44$, $\alpha = 0.8$ and time step $\Delta t = 0.1$

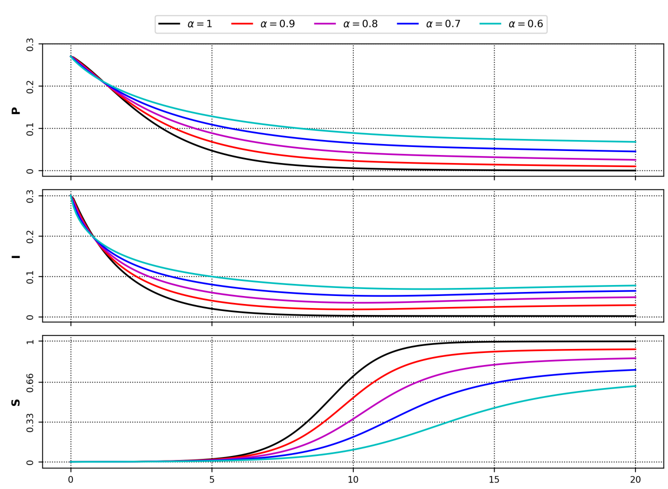
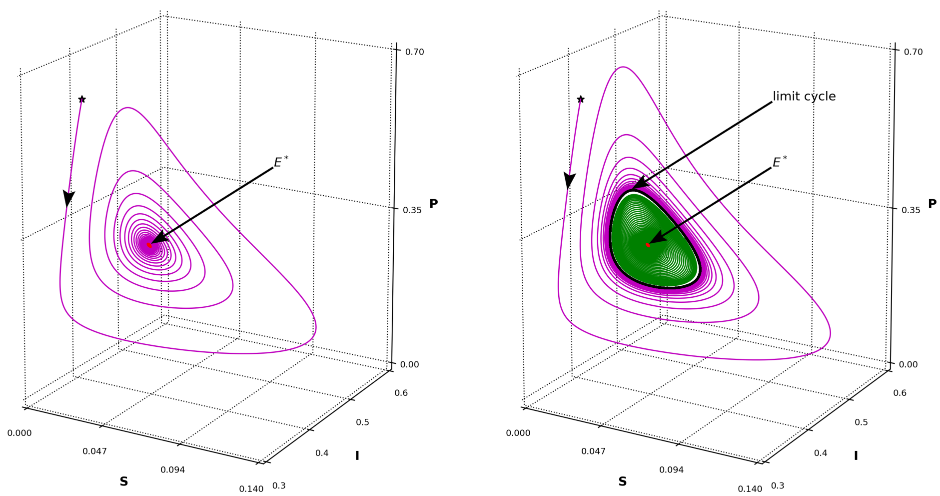


Figure 3: Time series of system (2) with various of α values by using parameter: $\beta = 0.51$, $\delta = 0.52$, $\mu = 0.01$, $\eta = 0.13$, $\omega = 0.92$, $\zeta = 0.44$, $\alpha = 0.8$ and time step $\Delta t = 0.1$

Now, we set the parameter values as follows: $\beta = 0.51$, $\delta = 0.01$, $\mu = 0.01$, $\eta = 0.13$, $\omega = 0.55$, $\zeta = 0.44$ and $\alpha = 0.8$. Thus we have three equilibrium points i.e. two saddle points E_0 and E_1 and a predator extinction point $E_3 = (0.020, 0.649, 0)$. According to the Theorem 2.1.(iii) and 6.3, equilibrium point E_3 is asymptotically stable, both locally and globally, see Figure 2a. Now, we increase the ratio of biomass conversion of susceptible prey parameter ω to $\omega = 0.92$, the asymptotically stable focus co-existence equilibrium point $E^* = (0.025, 0.475, 0.258)$ appears (see Theorem 5.2), and E_3 becomes a saddle point as shown in the Figure 2b. This condition shows that the predator population becomes extinct, and both of infected and susceptible prey populations exist. For the next simulation, we take parameters as in the first simulation, and show that when the order-α approaches to $\alpha = 1$, the solution of CFO system approaches the solution of the first order system, see Figure 3.



(a) $\alpha = 0.82$ (b) $\alpha = 0.84$

Figure 4: 3-D Phaseportraits of system (2) with parameter $\beta = 0.51$, $\delta = 0.01$, $\mu = 0.01$, $\eta = 0.13$, $\omega = 0.92$, $\zeta = 0.44$ and time step $\Delta t = 0.1$

Next, we show numerically that the stability of equilibrium point is also influenced by order of fractional derivative. For that, we choose parameter values as follows: $\beta = 0.51$, $\delta = 0.01$, $\mu = 0.01$, $\eta = 0.13$, $\omega = 0.92$ and $\zeta = 0.44$. If $\alpha = 0.82$ is replaced by $\alpha = 0.84$, the locally asymptotically stable point $E^* = (0.025, 0.475, 0.258)$ changes its stability and a stable limit cycle occurs simultaneously (see Figure 4b). This phenomenon is called Hopf bifurcation where the bifurcation point is $\alpha^* \approx 0.84257$.

9. CONCLUSION

We have discussed an eco-epidemiological fractional-order model that describes the interaction between predator and prey population with infectious diseases in prey. We have shown that this eco-epidemiological model has at most five biological equilibrium points, where the local and the global stability are completely analyzed. One of the expected conditions is that the extinction of infected prey population is achieved if the ratio of biomass conversion of susceptible prey is greater than the death rate of predator, and $\eta > \frac{\eta+\beta}{\mu+\delta}\zeta$ (locally stable) or $\frac{\beta\zeta-\delta\eta}{\eta-\zeta} < \frac{\beta\omega}{(1+\beta)\eta} < \mu$ (globally stable). We also prove that there is a condition when the fractional-order of derivative is varied, the stable focus co-existence point becomes unstable focus and isolated by a stable limit cycle, which is called a Hopf bifurcation. It means that all populations still exist along $t \to \infty$, but the density changes periodically. To illustrate the analytical results, we present numerical simulations using hypothetical parameter values. The application of our model to real data can be an interesting topic for future research.

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